ON THE CRLB FOR SOURCE LOCALIZATION IN A LOSSY ENVIRONMENT

B. Moussakhani, J.T Flåm, T. Ramstad*

Department of Electronics and Telecommunications
NTNU
N-7491 Trondheim, Norway

Ilangko. Balasingham

Interventional Center
Rikshospitalet University Hospital
N-0027 Oslo, Norway

ABSTRACT

In this work, localization of a source within a lossy medium is considered. By a lossy medium we mean an environment where the signal power decays exponentially with distance. We assume no prior knowledge on the source location, but that the source is heard by nearby sensors when transmitting. The source position shall be estimated based on the power received by these sensors. Under these assumptions, our focus is to determine the Cramer-Rao Lower Bound (CRLB). Thus, we are not studying specific estimators, but rather the (theoretical) performance of an optimal one. We demonstrate that the CRLB greatly depends on the shadowing conditions, and also on the relative positions of the sensors and the source. This spatial variability of the CRLB is used to discuss favorable positioning of the sensors.

1. INTRODUCTION

Generally, ultrasound waves have been used for communication and detection within the human body or in other environments where electromagnetic waves decay exponentially. However, new technologies in wireless communication, such as ultra-wideband, enable recovery of very weak signals and take advantage of the available bandwidth. Implant sensors and rescue robots are just two examples of the new trend. Although bandwidth requirements and the mobility of devices may vary for different applications, localization is considered a dominant requirement. As an example, consider a capsule endoscope which travels through the digestive system. Each picture taken by the endoscope is obviously more valuable if it can be associated with a precise location. The problem of obtaining these locations is in fact the main motivation for this paper. However, the results that we obtain apply for localization in any environment with exponential pathloss. Localization in free space based on Received Signal Strength (RSS) has been studied vastly for different kinds of waveforms and under different scenarios of shadowing and fading [1, 2], but to the best of our knowledge little been done for the environments with exponential pathloss. This paper investigates performance bounds on RSS based localization in media with exponential pathloss. Localization under three different shadowing scenarios is considered; in the first scenario, we assume the shadowing has a constant variance which is independent of the distance between the source and the receiver sensor. In the second scenario distance dependent shadowing is considered. Finally, distance dependent and spatially correlated shadowing is assumed. The CRLB, and its spatial variability, is quite different for these cases, underlining that the accuracy of an optimal estimator strongly depends on the shadowing conditions and on the positions of the source and sensors. Also we present the space correlation coefficient as an informative factor when deciding the number and positions of the sensors.

In the next section we introduce the system model. In section 3, the CRLB for three different shadowing scenarios is derived. The CRLB as a function of sensor density is studied in section 4 and finally we conclude the paper in section 5.

2. SYSTEM MODEL

A material which has an imaginary part in its permittivity ($\epsilon = \epsilon' + j\epsilon''$) or a nonzero conductivity ($\sigma \neq 0$) is called an absorbing material throughout this paper.

Solving the Helmholtz’s equations [3] for this kind of material will have the following solution for the electric field:

$$E_x = E_0 e^{-\alpha z} e^{-j\beta z}.$$  (1)

In (1), $\beta$ determines the phase constant, and $\alpha$ is the absorbing coefficient. The same equation also holds for the magnetic field, so the power decays with $2\alpha$, [4]. Considering the fact that the signal is received through many layers with different (random) absorption, a lognormal shadowing is expected for the received power [1]:

$$P_r = P_t e^{-2\alpha d S},$$  (2)

$$f_S(S, \mu, \sigma) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{\ln S - \mu^2}{2\sigma^2}}.$$
Here $P_r$ is the received power at the receiver which is located $d$ meters from a source whose power is $P_s$. The received power is subject to lognormal shadowing $S$ with $\mu = 0$. This pathloss model is also used in [1]. Note that it neglects the additive thermal noise in the receiver\(^1\). Expressing (2) in dB and substituting $2\alpha$ with $m$ gives

$$p_r = p_t - md + s = p_t - md - s,$$  

where the last equality holds because $s \sim N(0, \sigma)$. Equation (3) is the much used pathloss model for any absorbing environment [4], [5], where $m$ is the average pathloss coefficient. For a scenario where a group of sensors receive the signal coming from one source we can write the pathloss as

$$p_i = p_t - md + s.$$  

Here the vector $d$ is the distance between the source and each of the receiving sensors. Our goal is to find the Cramer-Rao Lower Bounds (CRLBs) for estimating the location of the source using the received powers.

### 3. LOCALIZATION LOWER BOUND

We calculate the CRLB for three scenarios based on three different shadowing scenarios.

#### 3.1. Shadowing with constant Variance

In the simplest scenario all sensors are subject to i.i.d. shadowing with constant variance, that is $s \sim N(0, \sigma^2)$. In (4). Without loss of generality and for simplicity we consider the two dimensional localization problem; the results can easily be extended to the three dimensional case. Having observed the power received by all $N$ sensors, and knowing the transmit power, the likelihood function can be written as

$$f_a(p_1 \mid d) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2\sigma^2}(p_i - md_i)^2}.$$  

Then according to the definition, the Fisher Information Matrix (FIM) for the vector $d$ can be found as

$$I_d = E \left[ \frac{\partial}{\partial d} \ln f_a(p_1 \mid d) \frac{\partial}{\partial d} \ln f_a(p_1 \mid d) \right] = \frac{m^2}{\sigma^2} I,$$

where $I$ is the $N \times N$ identity matrix. We are only interested in the location of the source, defined as $\theta = (x, y)$. So using the chain rule [6], the FIM for $\theta$ can be written as

$$I_\theta = HI_d H^T,$$

where

$$H = \begin{bmatrix} \frac{\partial d_1}{\partial x} & \frac{\partial d_1}{\partial y} & \cdots & \frac{\partial d_N}{\partial x} & \frac{\partial d_N}{\partial y} \end{bmatrix},$$

$$\varphi_i = \tan^{-1} \frac{y - y_i}{x - x_i},$$

Hence, the Fisher Information Matrix will be:

$$I_\theta = \frac{m^2}{\sigma^2} \begin{bmatrix} \sum_{i=1}^{N} \cos^2 \varphi_i & \cdots & \cos \varphi_i \sin \varphi_i \\ \cdots & \cdots & \cdots \\ \sum_{i=1}^{N} \sin \varphi_i & \cdots & \sum_{i=1}^{N} \sin^2 \varphi_i \end{bmatrix}.$$  

We define the overall CRLB as the trace of $I_\theta^{-1}$. Hence,

$$CRLB = \frac{N}{|I_\theta|} = \frac{N \sigma^2}{m^2 \sum_{i=1}^{N} \sum_{j=1}^{N} \cos \varphi_i \sin \varphi_j \sin(\varphi_j - \varphi_i)}.$$  

In (10) the CRLB is distance independent; it is only a function of how the sensors are placed around the source, while in free space it is known to be proportional to the distance [2]. The short explanation for this is that in free space power decays as $1/d^2$. This results in a FIM depending on $d$, whereas in our case, it does not - which can be seen in (6). In (10), minimizing the CRLB means to find the angles which maximize the denominator. Taking the derivative of the denominator in (10) w.r.t. $\varphi_i$ and setting it to zero gives

$$\sum_{j=1}^{N} \sin 2\varphi_j + \sum_{i=1}^{N} \sum_{j=1}^{N} \sin(\varphi_j - \varphi_i) = 0.$$  

In (11), for every $\phi_i - \phi_j$ there exists a negative angle $\phi_j - \phi_i$ and summing the sine functions for the two angles results in zero, so due to the symmetry the second term is always zero. Therefore minimizing the CRLB reduces to fulfilling the following condition

$$\sum_{j=1}^{N} \sin 2\varphi_j = 0.$$  

(12)

It is easy to see from (10) that every solution with $\varphi_1 = \varphi_2 = \cdots = \varphi_N$ minimizes the denominator. Also for $N > 2$, equally spaced angles $(\varphi_k = \frac{2\pi k}{N}, k = 0, 1, \ldots, N-1)$ solves (12), which we will see corresponds to a minimum for the CRLB. Figure (3.1) shows the CRLB for the case where the sensors are located on a circle with radius 1 meter, and $m = 1$. The plot results by moving the source around within this circular region, while keeping the sensors fixed on the circle. It is apparent that the CRLB varies with the position of the source.

\(^{1}\)When the source is a capsule endoscope, producing high quality video, the signal is received with acceptable SNR at the sensors. In addition it moves slowly. Thus, many power readings can be made at the sensors while the source is essentially in the same position. Consequently, the effects of (zero mean) thermal noise are averaged out.
the CRLB shows spatial variability. In this plot, with \( N = 4 \) sensors, the CRLB is smallest in the areas close to the center and increases as the source moves towards the circle border and it reaches the maximum at sensor locations. The reason is that the condition in (12) is easier to approximate for the areas closer to the center than the areas near the sensors. However, as we will see in section 4, by increasing the number of the sensors the spatial variability of the CRLB approaches zero.

### 3.2. Distance Dependent Shadowing

For the second scenario we consider shadowing where the variance is a function of distance, that is

\[
s \sim N(0, C(d)), \quad (13)
\]

where \( C(d) \) is a diagonal matrix. How shadowing is related to distance can vary for different scenarios, but we expect the shadowing variance to increase with distance. The reason is intuitive since we are considering a medium with random inconsistency: by increasing the distance between the source and the sensor we expect more randomness. We limit the investigations to the case where the variance increases linearly with distance. Empirical studies show that for the human body such a model can be justified [4]. So, as before, the problem in (4) amounts to estimating the unknown vector, \( \hat{d} \), given \( p_1 \sim N(\mu(d), C(d)) \), where \( \mu(d) \) and \( C \) are

\[
\mu(d) = md, \\
C = diag(Kd_1, Kd_2, \ldots, Kd_N). 
\]

We find the FIM for \( d \) by using the following equation [6] for each element \( I_d(i,j) \):

\[
I_d(i,j) = \frac{\partial \mu}{\partial d_i}^T C^{-1} \frac{\partial \mu}{\partial d_j} + \frac{1}{2} tr \left( C^{-1} \frac{\partial C}{\partial d_i} C^{-1} \frac{\partial C}{\partial d_j} \right). 
\]

In order to calculate (15) we have

\[
\frac{\partial \mu}{\partial d_i} = [0, \ldots, 0, \underbrace{m}_{\text{ith element}}, 0, \ldots, 0]^T, \\
\frac{\partial C}{\partial d_i} = diag(0, \ldots, 0, \underbrace{K}_{\text{ith element}}, 0, \ldots, 0).
\]

Substituting these results into (15) each element of the FIM is

\[
I_d(i,i) = \frac{m^2}{Kd_i^2} + \frac{1}{2d_i^2} \quad \text{and} \quad I_d(i,j) = 0. \quad (16)
\]

Applying the chain rule for (16) the FIM for estimating the source coordinates becomes

\[
I_\theta = \begin{bmatrix}
\sum I_d(i,i) \cos^2 \varphi_i & \sum I_d(i,i) \sin \varphi_i \cos \varphi_i \\
\sum I_d(i,i) \sin \varphi_i \cos \varphi_i & \sum I_d(i,i) \sin^2 \varphi_i
\end{bmatrix}.
\]

Taking the trace of the inverse we get

\[
CRLB = \frac{\sum I_d(i,i)}{|I_\theta|}. \quad (17)
\]

\[
|I_\theta| = \sum_{i=1}^{N} \sum_{j=1}^{N} I_d(i,i) I_d(j,j) \sin \varphi_j \cos \varphi_i \sin(\varphi_j - \varphi_i).
\]

The fact that (17) depends on both the angle and the distance between source and the sensor makes it intractable to find the best configuration around the source for the minimum CRLB. However, since the diagonal elements of \( I_d \) are inversely proportional to the distance, one would expect that a sensor may estimate its distance to the source with increased accuracy when the distance decreases. This is exactly what can be seen in Figure (3.1): areas close to the sensors have lower CRLBs, and the lowest points correspond to the sensor locations. This is in complete contrast to the previous scenario where the areas close to the sensors have the highest CRLB. Furthermore, the spatial variability of the CRLB appears to have increased compared to the previous scenario. This is indeed the case, and it will be shown in section 4.

### 3.3. Spatially Correlated and Distance Dependent Shadowing

As the third scenario we consider the case where the shadowing variance is a function of distance and in addition it is spatially correlated. So the problem is defined as in (4), but this time the covariance matrix \( C \) is not diagonal. As the correlation model, we used the widely accepted Gudmundson’s model [7], where the shadowing caused by the environment is exponentially decaying. Specifically,

\[
C(i,j) = \frac{\sqrt{G(i,j)}}{\delta} e^{-\frac{|d_{ij}|}{\delta}} \quad (18)
\]
where \( d_{i,j} = \| \theta_i - \theta_j \| \) is the Euclidian distance between sensor \( i \) and sensor \( j \), \( \delta \) is the space correlation coefficient and \( c_{ii} \) are the diagonal elements of the matrix \( C(d) \) in (13). Then we use equation (15) to derive the FIM for the vector \( d \). Since the covariance matrix is not diagonal, the corresponding elements of \( I_{d} \) and its determinant are hard to calculate in closed form. Numerically, however, they may always be calculated.

Figure (3.1) portrays the CRLB for \( \delta = 1 \) in the same setting as the last two cases, it can be seen that the behavior is almost like the second scenario, i.e. the CRLB reaches its minimum in the areas close to the sensors and it reaches its maximum along the line between two neighboring sensors. However, it can be seen that the spatial variability of the CRLB has further increased.

4. COMPARISON

Now we investigate how the CRLB is affected when the number of sensors in the network increases. So we consider sensors around a circle in the same setting as figure (1) and we vary the number of sensors between 3 to 50. Figure (2) shows, in logarithmic scale, how the mean value of the CRLB behaves under three different shadowing scenarios. For the constant shadowing case (Case1) we use \( m = 1 \) and \( \sigma^2 = 2 \). For the cases where mean and the variance increase linearly with distance (Case2 and Case3), we use \( m = 5 \), \( K = 6 \). For the Case3 two different space correlation coefficient are investigated \( \delta = \{ 0.1, 1 \} \). We can see that for Case1 and Case2 the CRLB decreases with a steeper slope than for Case3. The reason is that in Case3, by adding more sensors to the network, observations become more and more correlated - or said differently, more and more redundant. Increasing \( \delta \) from 0 (Case2) to 0.1, degrades the performance, especially for a large number of sensors. The same trend is more obvious when \( \delta = 1 \). Then, the curve is distinctively above the uncorrelated case.

Figure (3) shows the spatial variability of the CRLB for the same scenario as figure (2). It is clear that in Case1, it approaches zero with a very steep slope. This can be explained by the limiting case where the sensor density on the circle goes to infinity. Then, wherever the source is located within the circle, for any pair of neighboring sensors it sees equal angular separation. Hence the condition in (12) becomes approximately true for all points inside the circle. In contrast, the curves for Case3 stop falling after a limited number of sensors. For \( \delta = 0.1 \) there is even a slight increase in the spatial variability by increasing the number of sensors. This observation, together with the fact that the mean CRLB is decreasing for both of correlated cases in figure (2), implies that either the CRLB does not decrease further for some of the areas within the circle, or that for those areas the CRLB drops slower than for the surroundings, when adding sensors.

Next we investigate the effect of, not surrounding the source by sensors, but rather condensing the sensors on a circular arc (a segment of the the circumference). For tracking a capsule endoscope this is interesting, because it will indicate whether or not it is sensible to, for example, place all sensors on the abdomen. Figure (4) shows how the CRLB for the different shadowing cases behaves when we gradually reduce the arc size from a full circle to 0.1\( \pi \). For this study we use \( N = 12 \) sensors. The results show, even for Case1, that reducing the arc results in poorer performance. The reason is that the condition (12) is fulfilled for less and less points inside the circle. The figure also shows that for both Case2 and Case3 the CRLB increases by decreasing the arc. Meanwhile for Case3 with \( \delta = 0.1 \), the degradation in the CRLB is more than Case2 and that is because by decreasing the arc we are putting the sensors closer to each other - thus making the received powers more correlated. The same trend is even more pronounced for \( \delta = 1 \). From figure (4) we draw the conclusion that in order to increase the estimation accuracy we have to (i) place the sensors as far as possible from each other (this means that the full arc must be used) and (ii) that they are located as closely to the source as possible. These results are in line with the conclusions found in [8].
5. CONCLUSION

In this paper, the CRLB for localization within an absorbing medium under three different shadowing scenarios has been presented. The results show that the CRLB behaves differently for each of the scenarios. We have studied how changing the configuration of the sensors affects the CRLB, and showed that both distances and angles between the source and the sensors influence the CRLB. The rule of thumb is that the lowest CRLB is achieved when the sensors are as close as possible to the source and as far as possible from each others.

6. REFERENCES


