A Market-Clearing Model for Spectrum Trade in Cognitive Radio Networks

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ABSTRACT

We model cognitive radio networks (CRNs) as a spectrum market where every primary user (PU) offers her subchannels with certain interference bound indicating the interference limit the PU can tolerate, and secondary users (SUs) purchase the right to access the subchannels while observing their budget constraints as well as the interference bound. In this spectrum market model, the utility of SU is defined as the achievable transmission rate in free space, and the utility of PU is given by the net profit the PU can make. Then we develop a market equilibrium in the context of Fisher model, and show that the equilibrium is obtained by solving an optimization problem called Eisenberg-Gale convex program. Furthermore, we develop a distributed algorithm with best response dynamics and price dynamics, and prove that its asymptotic solutions are equivalent to the solutions given by the convex program. Besides, we introduce adaptive step size to the price dynamics for faster convergence. With some numerical examples, we show that it helps to achieve faster convergence.

Categories and Subject Descriptors
C.2.1 [Computer-Communication Networks]: Network Architecture and Design—Distributed networks; Wireless communication; G.1.6 [Optimization]: Convex programming

General Terms
Algorithms, Design, Economics, Theory

Keywords

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MobiHoc’11 May 16-19, 2011, Paris, France
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1. INTRODUCTION

Recently, market-based approaches have started being deployed to various scenarios of cognitive radio network (CRN) since the behaviors of the wireless users in CRN can be cast into those of the traders in general market (i.e., maximizing their utilities or profits), and the equilibrium state, called market equilibrium, complies with the key requirement of the CRN, i.e., efficient utilization of spectral resources.

In the market-based approaches, spectra and interference are regarded as marketable commodities. In words, primary users (PUs) offer their licensed spectra with collecting certain monetary rewards from transmitting secondary users (SUs); SUs purchase the offered spectra by adjusting their transmission powers. Like a general market, every PU has a limitation in her production (i.e., spectra and interference), and every SU has budget constraint.

In this work, we consider a multi-channel sharing CRN where the frequency range is divided into multiple subchannels, and each subchannel is orthogonally licensed to a single PU. Defining the utilities of SUs and PUs as the achievable transmission rates and the net profits the PUs make, respectively, we study a market equilibrium in terms of Fisher model, which results in the market clearance and maximum utility of every trader. The equilibrium in the Fisher model is achieved by solving a convex program called Eisenberg-Gale convex program whose objective is to maximize the log-joint utility function of all SUs over a convex region defined via a set of linear constraints [4]. We show that (i) the solution of the convex program satisfies the weak Pareto optimality in SUs’ utilities, (ii) the Lagrangian dual variables turn out to be the equilibrium price, and (iii) the equilibrium price also makes the PUs’ utilities core-stable.

We also consider a distributed implementation of the convex program, which enables PUs and SUs to decide their strategy without any central authority: given the price of each subchannel, each SU computes her optimal transmission power vector using best response dynamics, and each PU updates the price of each subchannel she owns through linear dynamics (we call it price dynamics). We show that the price dynamics is asymptotically stable with any initial points, and the solutions yielded at the stable state is equivalent to the solutions given by the convex program (i.e.,

1Henceforth, SU implies transmitting SU.
market equilibrium). Furthermore, we apply schemes with adaptive step size to the price dynamics in order to accelerate the convergence. We conceive two different schemes: adaptive decrease only and adaptive increase and decrease. By numerical evaluations, we illustrate that the adaptive step size is usually beneficial for reducing the number of iterations to the convergence.

The rest of this paper is organized as follows:

In Section 2, we survey some recent related work. In Section 3, we give our spectrum market model. In Section 4, we define the equilibrium in our spectrum market. In Section 5, we give the Eisenberg-Gale convex program and investigate its properties. In Section 6, we present the distributed algorithm for solving the convex program, and analyze its stability and asymptotic optimality. In Section 7, we give several results of numerical evaluations. We conclude this paper in Section 8.

2. RELATED WORK

Recently, several research efforts have deployed the market-based approach to CRNs. Xing et al. [15] consider a spectrum market where different consumers may evaluate the same supplier differently according to their applications and locations. Considering limited information, they develop price dynamics with a stochastic learning algorithm in order to find the optimal price yielding maximum benefit of the suppliers.

Hong et al. [3] propose a fully distributed algorithm - no collaboration among SUs and PUs - that achieves a market equilibrium in multi-channel sharing CRN such that the supply of the spectrum equals to its demand, and the network of SU is stable. They investigate and present the convergence condition of the algorithm in terms of the power gain (i.e., channel gain).

Niyato et al. [8] model a multi-level bandwidth sharing in CRN into an interrelated market. They propose an adaptive decrease only function. Then they show that the market equilibrium is given by the solution of a linear complementarity problem, and under the symmetric channel gain and low-rank conditions, they prove that this problem becomes equivalent to the problem of finding KKT points of a quadratic program. Furthermore, they develop a decentralized tatonnement process that converges to the equilibrium. However, they do not include the manager’s utility in the market equilibrium, and assume that each channel has fixed bound of tolerable interference. Moreover, the KKT points of the quadratic program do not guarantee the optimality, and because of this reason, it is not verified whether the distributed algorithm (tatonnement process) converges to the optimal equilibrium even asymptotically.

3. SPECTRUM MARKET MODEL

We consider a CRN where all PUs and SUs are located within a limited geographical region. Then we address the spectral resource on the frequency domain taking the multi-channel sharing into account - that is, the whole frequency range is divided into multiple subchannels. In this work, we premise that each subchannel is exclusively licensed to a single PU; it however can be shared with multiple SUs concurrently unless the SUs invoke interference larger than the PU can tolerate.

Prior to giving the market model, we define the following denotations: 1) \( I \): Set of transmitting SUs; 2) \( \mathcal{L} \): Set of PUs, and we let \(|\mathcal{L}| = m\); 3) \( J \): Set of subchannels; 4) \( u_i: \mathbb{R}^n \to \mathbb{R} \): Utility function of SU \( i \in I \) where \( n = |J| \); 5) \( \mathbf{p} = [p_{l1}, \ldots, p_{ln}]^T \): Transmission power vector of SU \( i \); 6) \( p_{ij} \): SU \( i \)'s transmission power on subchannel \( j \in J \); 7) \( J_l \): Set of subchannels licensed to PU \( l \in \mathcal{L} \), and \( J_l \cap J_l' = \emptyset \) if \( l \neq l' \); 8) \( v_l: \mathbb{R}^n \to \mathbb{R} \): Utility function of PU \( l \); 9) \( \pi_{ij} \): Price marked by PU \( l \) on subchannel \( j \); 10) \( \pi_l = [\pi_{1l}, \ldots, \pi_{nl}]^T \): Price vector of PU \( l \); 11) \( y_l = [y_{l1}, \ldots, y_{ln}]^T \): Vector of the interference offered by PU \( l \); 12) \( y_j \): Amount of interference on subchannel \( j \) offered by PU \( l \).

Now we develop the spectrum market model with the following key considerations:

1. Subchannel and interference are interpreted as the type of commodity and the quantity of each commodity, respectively.
2. The price is marked on every subchannel, and given as a price per unit interference.
3. The utility of SU is defined as the summation of the rate achievable on every subchannel, that is,

\[
u_i (\mathbf{p}_i) = \sum_{j \in J} B_j \log_2 \left( 1 + \frac{p_{ij} G_{ij}}{N_0 + \Gamma_j} \right) \tag{1}\]

where \( B_j \) is the bandwidth of subchannel \( j \in J \), \( G_{ij} \) is the channel gain on subchannel \( j \) between SU \( i \) and her target, \( \Gamma_j \) is the interference invoked from the PU who owns subchannel \( j \), and \( N_0 \) is the thermal noise. As shown in (1), the utility of SU does not reflect the interference among themselves\(^2\).
4. SUs cannot purchase the interference larger than PUs offer. That is, \( \forall l \in \mathcal{L} \) and \( \forall j \in \mathcal{J}_l \),
\[ \sum_{i \in \mathcal{I}} p_{ij} G_{ij}^l \leq y_{ij} \quad (2) \]
where \( G_{ij}^l \) is the channel gain on subchannel \( j \) between SU \( i \) and PU \( l \).

5. Each SU \( i \) has an endowment of budget \( e_i > 0 \), and the total budget spent on the purchase of subchannels cannot exceed \( e_i \). That is, \( \forall i \in \mathcal{I} \),
\[ \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}_l} \pi_{ij} p_{ij} G_{ij}^l \leq e_i \quad (3) \]
6. Each PU \( l \) offers her subchannels, i.e., subchannels in \( \mathcal{J}_l \), to SUs with price \( \pi_{ij} \), and her utility function \( v_l \) is defined as the net profit she makes, i.e.,
\[ v_l = \sum_{j \in \mathcal{J}_l} \pi_{ij} y_{ij}. \quad (4) \]

It is assumed that, while trading, PUs are concerned with only making maximal profits on the sales of the subchannels.

7. Every subchannel can be accessed by one or more SUs. Therefore we let every PU \( l \) set an upper bound \( Y_l \) such that the total offered interference over \( \forall j \in \mathcal{J}_l \) should not exceed the bound. That is, \( \forall l \in \mathcal{L} \),
\[ \sum_{j \in \mathcal{J}_l} y_{ij} \leq Y_l. \quad (5) \]

We let the channel gain reflect the free space path loss defined by Friis transmission equation given in [11]. Then the higher frequency a subchannel has, the more amount of information a user can transmit on it given a fixed transmission power and subchannels with equal bandwidth [15]. In order to let every subchannel have equal opportunity of being purchased, we assume that the frequency range is divided into subchannels in the way that every subchannel yields similar transmission rate as possible given a constant transmission power and separate distance.

4. **MARKET EQUILIBRIUM**

Henceforth, we let \( G_{ij}^l = 0 \) for all \( l \in \mathcal{L} \) and \( j \notin \mathcal{J}_l \) for the simplicity. Based on the key considerations mentioned in Section 3, we define the equilibrium in the spectrum market following Fisher model [4]: the equilibrium is defined as a non-negative price vector \( \pi = [\pi_1, \ldots, \pi_m]^T \) at which there exist a transmission power vector \( \mathbf{p} \), for each SU \( i \), and a vector of offered interference \( \mathbf{y}_i \) for each PU \( l \) such that the following conditions hold:

1. For each SU \( i \), \( \mathbf{p} \), maximizes \( u_i \) over all \( \mathbf{p} \in \mathbb{R}^{m} \) such that \( \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \pi_{ij} p_{ij} G_{ij}^l \leq e_i \).

2. For each PU \( l \), the vector \( \mathbf{y}_l \) maximizes \( v_l \) subject to \( \sum_{j \in \mathcal{J}_l} y_{ij} \leq Y_l \).

3. For each subchannel \( j \), \( \sum_{i \in \mathcal{I}} y_{ij} = \sum_{i \in \mathcal{I}} \sum_{l \in \mathcal{L}} p_{ij} G_{ij}^l \).

4. \( \sum_{i \in \mathcal{L}} \sum_{j \in \mathcal{J}} \pi_{ij} y_{ij} = \sum_{i \in \mathcal{I}} e_i \).

Equilibrium price in the Fisher model is also known as *market clearing price* since it clears not only all the commodities offered by suppliers but also all the budget possessed by consumers; that is, it clears all the budget possessed by SUs as well as all the interference offered by PUs.

5. **EISENBERG-GALE CONVEX PROGRAM**

In this section, we develop Eisenberg-Gale convex program (henceforth, convex program) [4, 2], that yields the market equilibrium defined in 4. We notice that the convex program is established in terms of SUs’ utility functions.

### 5.1 Monotone-Transformation of The Utility Function

The convex program yields the market equilibrium only when the utility functions are concave and *homogenous of degree one*. As given in (1), the utility function is concave over \( \mathbf{p} \in \mathbb{R}^m_+ \) but not homogeneous of degree one. Therefore, prior to presenting the convex program, we transform the utility function of SU into a function homogeneous of degree one by the following theorem [4]:

**Theorem 1.** Let \( u : \mathbb{R}^m_+ \rightarrow \mathbb{R} \) be a continuous, strict monotonic, concave, and homothetic function. Then there is a monotone-transformation yielding a function \( f : \mathbb{R}^m_+ \rightarrow \mathbb{R}_+ \) that is homogeneous of degree one, and preserves continuity, strict monotonicity, concavity, and homotheticity, and satisfies:

1. If \( u (\mathbf{p}) = 0 \), then \( f (\mathbf{p}) = 0 \).
2. If \( u (\mathbf{p}) \neq 0 \), then there exists a unique \( \alpha \in \mathbb{R}_+ \) such that \( u (\mathbf{p}/\alpha) = 1 \), and \( f (\mathbf{p}) = \alpha \).

### 5.2 The Convex Program

By the monotone-transformation, we can transform \( u_i \) to \( f_i \) that is homogeneous of degree one. Then we develop the convex program that yields the market equilibrium as follows:

\[ \text{maximize} \quad \sum_{i \in \mathcal{I}} e_i \ln (f_i) \quad (6) \]
subject to
\[ \sum_{j \in \mathcal{J}_l} y_{ij} \leq Y_l, \quad \forall l \in \mathcal{L}; \quad (7) \]
\[ \sum_{i \in \mathcal{I}} \pi_{ij} G_{ij}^l \leq y_{ij}, \quad \forall l \in \mathcal{L} \quad \text{and} \quad \forall j \in \mathcal{J}; \quad (8) \]
\[ p_{ij} \geq 0 \quad \forall i,j. \quad (9) \]

As shown in the above convex program, we allow negative \( y_{ij} \), which implies that, if \( y_{ij} < 0 \), then the subchannel \( j \) is being used excessively by the PU who owns it. The solution of the convex program is often regarded as Nash bargaining solution with zero disagreement point. Therefore it has a unique solution vector that satisfies weak Pareto optimality due to the strict concavity of the objective function and linearity of the constraints [7].

Let \( \tilde{\mathbf{p}} = [\tilde{p}_1, \ldots, \tilde{p}_m] \) and \( \tilde{\mathbf{y}} = [\tilde{y}_1, \ldots, \tilde{y}_m] \) denote the optimal solutions to the convex program. Since \( f_i (\tilde{\mathbf{p}}) > 0 \) for
all \(i\), we have the following KKT optimality conditions with the corresponding Lagrangian multipliers \(\eta\) and \(\pi_{ij}\):

\[
\tilde{p}_{ij} \left( \frac{e_i}{f_i(p_{ij})} \frac{\partial f_i(p_{ij})}{\partial p_{ij}} - \sum_{l \in \mathcal{L}} \pi_{ij} G_{ij}^l \right) = 0 \quad \forall i, j, \quad (10)
\]

\[
e_i \frac{\partial f_i(\tilde{p}_{ij})}{\partial p_{ij}} - \sum_{l \in \mathcal{L}} \pi_{ij} G_{ij}^l \leq 0 \quad \forall i, j. \quad (11)
\]

\[
\eta_i - \pi_{ij} = 0, \quad \forall l, j, \quad (12)
\]

\[
\eta_i \left( \sum_{j \in \mathcal{J}} \tilde{y}_{ij} - Y_i \right) = 0, \quad \forall l, \quad (13)
\]

\[
\pi_{ij} \left( \sum_{j \in \mathcal{J}} \tilde{p}_{ij} G_{ij}^l - \tilde{y}_{ij} \right) = 0, \quad \forall l, j, \quad (14)
\]

and the constraints (7) and (8).

By (12), we notice that, for every PU, marking an identical price vector \(\eta\) optimizes the utility of every PU.

By (10), (12) and Euler’s theorem [12], the following equation holds as well:

\[
\sum_{j \in \mathcal{J}} \eta_{ij} G_{ij}^l = \frac{e_i}{f_i(p_{ij})} \sum_{j \in \mathcal{J}} \frac{\partial f_i(p_{ij})}{\partial p_{ij}} \tilde{p}_{ij} = e_i \quad \forall i \quad (17)
\]

implies that each SU spends her budget completely under prices \(\eta_1, \ldots, \eta_m\).

Next we show that the market clears. By (17) and (14), we get:

\[
\sum_{j \in \mathcal{J}} \eta_{ij} \sum_{l \in \mathcal{L}} \tilde{p}_{ij} G_{ij}^l = \sum_{l \in \mathcal{L}} \sum_{j \in \mathcal{J}} \eta_{ij} \tilde{y}_{ij} \quad \text{by (17)}
\]

\[
= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \tilde{y}_{ij} - Y_i \quad \text{by (14)}
\]

\[
\sum_{j \in \mathcal{J}} \bar{y}_{ij} - Y_i = 0 \quad \forall l. \quad (22)
\]

Eventually, we conclude that the price vector given by the Lagrangian multipliers is the equilibrium price that clears the market.

The equilibrium price can be computed by solving the system of the linear equations (10) ~ (14) that is however normally inconsistent. Thus we need to provide a certain precision bound on each linear equation at least in order to achieve an approximate equilibrium price.

Besides, in order to solve the system of the linear equations, we need to compute the partial derivatives of the monotone-transformed function (required in (10) and (11)), and it is given in [4]: if we let \(f(p) = \alpha\), then

\[
\frac{\partial f(p)}{\partial p_{ij}} = \alpha \frac{\partial u(p/\alpha)}{\partial p_{ij}} \frac{p}{u(p/\alpha)^T} p \quad (24)
\]

### 5.3 Core Stability of The Production Vector

In this section, we show that the solution \(\bar{y}_{ij}\) yielded by the convex program is in the core of NTU (non-transferable utility) game in the context of cooperative game theory [9].

In a cooperative game, the existence of non-empty core guarantees that no player will break away from the grand coalition (i.e., the cooperation of all the players) since the payoffs achieved by the cooperation within any subcoalition cannot be larger than the payoff yielded by the cooperation of the grand coalition. Therefore, if the solution \(\bar{y}_{ij}\) yielded by the convex program is in the core of the spectrum market, we guarantee that no PU will leave the market.

Denoting the core of the spectrum market as \(C(V)\), the core of an NTU game is defined as:

\[\text{Definition 1. For a } S \subseteq \mathcal{L}, \text{ let } V(S) = \{v_i(y_i) : l \in S\}, \text{ and } y = [y_1, \ldots, y_m]^T. \text{ Then the core of the spectrum market, } C(V) \text{ is defined as the set of all undominated imputations, i.e., } \bar{y} \in C(V), \text{ if and only if there is no } S \subseteq \mathcal{L}, S \neq \emptyset, \text{ and } y \text{ such that } v_i(y_i) > v_i(\bar{y}) \text{ for all } l \in S.\]

Then the following theorem holds:

**Theorem 3.** The interference solution \(\bar{y}\) of the convex program belongs to \(C(V)\).
6. DISTRIBUTED ALGORITHM

We also develop a distributed implementation whose stationary point is equivalent to the optimal solution given by the convex program.

6.1 The Algorithm

A natural class of dynamics in multiplayer noncooperative system is the best-response dynamics where each player updates her strategy to maximize her utility given the strategies of other players [1]. In this algorithm, each SU deploys the best response dynamics to maximize her own utility given the price vector and her budget constraint. Let \( \eta = [\eta_1, \ldots, \eta_m]^T \). Then, the best response of SU \( i \) is given by the power vector \( \beta_i \) such that

\[
\beta_i (\eta, y) = \arg\max_{\eta_i \sum_{j \in J} \tilde{y}_{ij} \leq y_i} e_i \ln f_i (p_i).
\] (30)

Accordingly, the algorithm with the best-response dynamics is given as follows:

for each \( l \in \mathcal{L} \) do
  Initialize the price \( \eta_l (0) \);
end for

\( t \leftarrow 0 \);

loop
  for each \( i \in \mathcal{I} \) do
    Find the best response \( \beta_i (t) \) using (30);
  end for

  for each \( l \in \mathcal{L} \) and \( j \in \mathcal{J} \) do
    Determine \( y_{ij} (t) \) such that
    \[
y_{ij} (t) = \sum_{i \in I} \beta_{ij} (t) G_{ij}.
    \] (31)
  end for

end loop

\( \{ \text{Each PU determines the amount of interference to be offered on each subchannel according to the current best responses of SUs.} \} \)

end for

for each \( l \in \mathcal{L} \) do
  Adjust the price by
  \[
  \eta_l = \delta \left( \sum_{j \in J} y_{ij} (t) - Y_l \right)_{+}.
  \] (32)
  \( \{ \text{Each PU updates her price in proportion to the amount of violations in her total interference constraint.} \} \)
end if

end loop

In this algorithm, \( \delta \) and \( \epsilon \) indicate the adjustment speed of the price and the termination condition of the algorithm, respectively. \( (a)^+ \) implies \( \max (a, 0) \) if \( b = 0 \), and equal to \( a \) if \( b > 0 \). The number of iterations required to reach the termination condition is strongly dependent on the size of the adjustment step (i.e., \( \delta \)).

6.2 Investigation of Asymptotic Equivalence to Convex Program

If the system of the linear equations (10) \( \sim (14) \) is consistent, and also if the price dynamics given by (32) is asymptotically stable, we can prove that the distributed algorithm yields the solutions asymptotically equivalent to the solutions of the convex program. That is, the solutions yielded at \( t = \infty \) is equivalent to the solutions of the convex program. In this subsection, we develop the proof of the asymptotic equivalence with postulating that the system of the linear equations (10) \( \sim (14) \) is consistent and the price dynamics are asymptotically stable.

THEOREM 4. Postulating that the system of the linear equations (10) \( \sim (14) \) are consistent, and also if the price dynamics are asymptotically stable, the solutions and equilibrium price obtained by the distributed algorithm are asymptotically equivalent to those yielded by the convex program and its KKT conditions.

PROOF. For every \( i \), at least one of the elements of \( \tilde{p}_i \) should be non-zero in order to make \( f_i > 0 \) hold. Moreover, due to the strict monotonicity of \( f_i \), the partial derivatives of \( f_i \) should be always positive. Thus \( \eta_l \) should be non-zero for all \( l \). Then the KKT conditions of the convex program are reduced to

\[
\bar{p}_{ij} \left( \frac{e_i}{f_i (p_i)} \frac{\partial f_i (p_i)}{\partial p_{ij}} - \sum_{i \in \mathcal{L}} \eta_i G_{ij} \right) = 0 \quad \forall i, j
\] (33)

\[
\frac{e_i}{f_i (p_i)} \frac{\partial f_i (p_i)}{\partial p_{ij}} - \sum_{i \in \mathcal{L}} \eta_i G_{ij} \leq 0 \quad \forall i, j
\] (34)

\[
\sum_{j \in \mathcal{J}} \tilde{y}_{ij} - Y_l = 0, \quad \forall l,
\] (35)
\[ \sum_{i \in I} \tilde{p}_{ij} G^i_{ij} - \tilde{y}_j = 0, \quad \forall i, j. \tag{36} \]

If we denote the best response of each SU \( i \) on \( t = \infty \) as \( \tilde{\beta}_i \) and the price of PU \( l \) on \( t = \infty \) as \( \tilde{\eta}_l \), then the following KKT conditions hold with additional Lagrangian multiplier \( \kappa_i \geq 0 \):

\[ \tilde{\beta}_{ij} \left( \frac{e_i}{f_i} \frac{\partial f_i}{\partial p_{ij}} \right) - \kappa_i \sum_{i \in I} \tilde{\eta}_l G^i_{ij} = 0 \quad \forall j \]  \( \tag{37} \)

\[ \frac{e_i}{f_i} \frac{\partial f_i}{\partial p_{ij}} \leq \kappa_i \sum_{i \in I} \tilde{\eta}_l G^i_{ij} \leq 0 \quad \forall j \]  \( \tag{38} \)

\[ \kappa_i \left( \sum_{j \in J} \sum_{i \in I} \tilde{\eta}_l \tilde{\beta}_{ij} G^i_{ij} - e_i \right) = 0. \tag{39} \]

Summing (37) over all \( j \), we get

\[ \sum_{j \in J} \frac{e_i}{f_i} \frac{\partial f_i}{\partial p_{ij}} \tilde{\beta}_{ij} = \kappa_i \sum_{i \in I} \sum_{j \in J} \tilde{\beta}_{ij} G^i_{ij}. \tag{40} \]

and, by Euler’s theorem,

\[ e_i = \kappa_i \sum_{i \in I} \sum_{j \in J} \tilde{\beta}_{ij} G^i_{ij}. \tag{41} \]

Substituting \( e_i \) of (39) with (41),

\[ \kappa_i \left( 1 - \kappa_i \right) \left( \sum_{j \in J} \sum_{i \in I} \tilde{\eta}_l \tilde{\beta}_{ij} G^i_{ij} \right) = 0. \tag{42} \]

For all \( i \), there should exist at least one \( j \) such that \( \tilde{\beta}_{ij} \neq 0 \) since \( f_i \left( \tilde{\beta}_i \right) > 0 \). Subsequently, in order to make both (37) and (42) hold for all \( i, \kappa_i = 1 \) since the partial derivatives of \( f_i \) are always positive. Therefore, the KKT condition (37) and (38) turn out to be identical with (33) and (34), respectively, if we let \( \tilde{\beta}_i = \tilde{p}_i \) for all \( i \) and \( \tilde{\eta}_l = \eta \) for all \( l \).

Let \( \tilde{y}_i = [\tilde{y}_1, \ldots, \tilde{y}_n]^T \) be the interference offer of PU \( l \) on \( t = \infty \). Then the following hold:

\[ \tilde{y}_j = \sum_{i \in I} \tilde{\beta}_{ij} G^i_{ij} \quad \forall l, j \]  \( \tag{43} \)

by (31), and

\[ \sum_{j \in J} \tilde{\eta}_j = Y_l \quad \forall l \]  \( \tag{44} \)

by (32).

If we let \( \tilde{y}_i = \tilde{y}_i \) for all \( l \) and \( \tilde{\beta}_i = \tilde{p}_i \) for all \( i \), (43) and (44) coincide with (35) and (36), respectively. Since the equilibrium price as well as the solutions of the convex program are unique, we confirm \( \tilde{y}_i = \tilde{y}_i \) and \( \tilde{\eta}_l = \eta \) for all \( l, \tilde{\beta}_i = \tilde{p}_i \) for all \( i \). Finally, we conclude that the solution and equilibrium price yielded by the distributed algorithm are asymptotically equivalent to those given by the convex program and its KKT conditions.

### 6.3 Stability Analysis

We prove the asymptotic stability of the price dynamics given in (32):

**Theorem 5.** The price dynamics of the distributed algorithm are globally asymptotically stable.

**Proof.** We consider the following Lyapunov function:

\[ V(\eta) = \frac{1}{2\delta}(\eta - \tilde{\eta})^T(\eta - \tilde{\eta}). \tag{45} \]

By differentiating the Lyapunov function, we get

\[ \dot{V}(\eta) = \sum_{i \in I} (\eta - \tilde{\eta})^T \sum_{j \in J} (\tilde{y}_j - Y_i) \]

\[ \leq (i) \sum_{i \in I} (\eta - \tilde{\eta})^T \sum_{j \in J} (\tilde{y}_j - Y_i) \]

\[ \leq (ii) \sum_{i \in I} (\eta - \tilde{\eta})^T \sum_{j \in J} \sum_{i \in I} (\tilde{p}_i G^i_{ij} - \tilde{\beta}_{ij} G^i_{ij}) \]

\[ = \sum_{i \in I} \sum_{j \in J} \sum_{i \in I} (p_{ij} G^i_{ij} - \tilde{\beta}_{ij} G^i_{ij}) \]

\[ = \sum_{j \in J} \sum_{i \in I} \sum_{i \in I} (\tilde{p}_{ij} - \tilde{\beta}_{ij}) \]

\[ \times \left( \frac{e_i}{f_i(p_j)} \frac{\partial f_i}{\partial p_{ij}} - \frac{e_i}{f_i(\tilde{\beta}_i)} \frac{\partial f_i}{\partial p_{ij}} \right) \]

\[ \leq 0 \tag{46} \]

where (i) follows from, if the projection \((\cdot)^+_{\eta_l}\) is not active, then the equality holds, and otherwise, the right-hand side of (i) is positive while \( V(\eta) \) is zero. The equality (ii) and (iii) follow from (43) and (37), respectively. Finally, the inequality (iv) holds due to the strict concavity and monotonicity of \( f_i \). Furthermore, \( V(\eta) \) is negative definite, and \( \|\eta\| \to \infty \) implies \( V(\eta) \to \infty \). Therefore, the \( \eta - \tilde{\eta} \) is globally asymptotically stable equilibrium point [5].

### 6.4 Deployment of Adaptive Step Size

We develop two heuristics that adapt the step size according to the behavior of the price dynamics. We anticipate faster convergence to the termination conditions.

1. Decrease the step size gradually if there are oscillations in the price dynamics, and maintain the current step size otherwise.

2. Decrease the step size gradually if there are oscillations in the price dynamics, and increase the step size gradually otherwise.

We name the first heuristic and the second heuristic NIAD (No Increase/Adaptive Decrease) and AIAD (Adaptive Increase/Adaptive Decrease), respectively. We evaluate these heuristics by numerical experiments with varying the amount of the increments and decrements in the step size.

### 7. Numerical Evaluations

#### 7.1 Experimental Setup

We generate a CRN within a 500m \( \times \) 500m square, and consider the frequency range of 54-862 MHz TV band following the IEEE 802.22 standard [18]. As mentioned in Section
3, we divide the frequency range into subchannels in the way that every subchannel yields equal transmission rate as possible given a constant transmission power. We vary the sizes of SUs, PUs, and subchannels according to the types of the experiments we perform. Besides, we use Interior Point Optimizer (IPOPT) [13] for solving the convex program and the best response dynamics. Additional experimental parameters are chosen as the following: 1) $\epsilon$: randomly chosen from $(0, 1.0]$; 2) $Y$: 8e-08 for all PUs; 3) Thermal noise: 1e-10; 4) PU’s transmission power: 0.1W for all PUs; 5) Initial price for the distributed algorithm: 6e06 per unit interference for all PUs; 6) Termination condition of the distributed algorithm: 0.05% $\times Y_i$.

### 7.2 Illustration and Evaluation of Equilibrium Price

First we measure the transition of the total interference demand according to the price change when the function (6) is maximized subject to the budget constraint (i.e., (3)). For these experiments, we locate 3 SUs and 2 PUs accommodating 8 subchannels. Setting $Y_i = 8e-08$, the measured results are shown in Fig. 1. It is observed that the demand decreases as the prices increase; by the definition, the equilibrium price is obtained when the total interference demand equals to $2 \times Y_i$, that is, 1.6e-07.

### 7.3 Evaluations of the Distributed Algorithm

We evaluate the distributed algorithm in terms of the convergence speed and the solution quality.

#### 7.3.1 Evaluations of the Convergence Speed

First, we illustrate the convergence process to the equilibrium point with the distributed algorithm. In this set of experiments, we consider 3 SUs, 3 PUs and 8 subchannels, and apply the constant adaptive size of 1e13. Fig. 2 illustrates the convergence trajectories of the utilities of SUs and PUs (Fig. 2a and Fig. 2b, respectively), and prices (Fig. 2c) as the iteration proceeds. The arrow lying along with each trajectory indicates the convergence direction. It is observed that, as the algorithm approaches to the equilibrium point, the amount of update in each iteration decreases, and which illustrates the asymptotic stability of the algorithm.

Then we locate 16 SUs and 16 PUs accommodating 64 subchannels, and measure the number of iterations required until the algorithm terminates. For the schemes with the adaptive step sizes, we set initial step size 1e14. Table 1 lists the measured results. As shown in this table, we can achieve the fastest convergence applying AIAD with the increasing ratio of 0.1 and the decreasing ratio of 0.2. Thus we conjecture that (i) both of the adaptive schemes yield better convergence speed than the scheme with a constant step size regardless of the size of the ratio, and (ii) the AIAD scheme outperforms the NIAD. However, we cannot confirm explicitly which adaptive scheme and values of the ratios yield the fastest convergence in general case. Currently, it is inevitable to find the best scheme and ratios by trial-and-error. Nonetheless, it may be possible to investigate an appropriate ratio theoretically given problem instances, and which remains as one of our future work.

#### 7.3.2 Evaluations of the Solution Quality

Finally, we evaluate the solutions and equilibrium price determined by the distributed algorithm. We perform these experiments with 8 SUs, 8 PUs and 32 subchannels.

First, we measure the absolute gaps between the initial budget of every SU and her payment given by solving the system of the KKT conditions (i.e., (33) ∼ (36)) with smallest allowable precision bound (measured 4.572e-06).
Table 1: Number of iterations until the termination. IR / DR implies increasing ratio / decreasing ratio.

<table>
<thead>
<tr>
<th>Adaptation scheme</th>
<th>IR / DR</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fixed as 1e12</td>
<td>-</td>
<td>841</td>
</tr>
<tr>
<td>Fixed as 1e13</td>
<td>-</td>
<td>418</td>
</tr>
<tr>
<td>Fixed as 1e14</td>
<td>-</td>
<td>Divergent</td>
</tr>
<tr>
<td>NIAD</td>
<td>- / 0.3</td>
<td>247</td>
</tr>
<tr>
<td>NIAD</td>
<td>- / 0.2</td>
<td>155</td>
</tr>
<tr>
<td>NIAD</td>
<td>- / 0.1</td>
<td>343</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.1 / 0.4</td>
<td>140</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.1 / 0.3</td>
<td>107</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.1 / 0.2</td>
<td>29</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.1 / 0.1</td>
<td>150</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.2 / 0.4</td>
<td>389</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.2 / 0.3</td>
<td>113</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.2 / 0.2</td>
<td>62</td>
</tr>
<tr>
<td>AIAD</td>
<td>0.2 / 0.1</td>
<td>80</td>
</tr>
</tbody>
</table>

measured results are plotted in Fig. 3a together with the results with the distributed algorithm. We see that the distributed algorithm yields larger gaps than the KKT conditions, and the largest absolute gap is measured 1.5e-4 at most.

With regards to SUs’ utilities and the equilibrium prices, we observe that the distributed algorithm and KKT conditions give almost identical values as shown in Fig. 3b and 3c.

Besides, we apply the solutions given by the distributed algorithm (i.e., $\beta$ and $\gamma$) to the KKT conditions in order to quantify the errors in the KKT conditions. For the purpose of comparison, we also measure the errors produced by the solutions of the convex program. Ideally, the KKT condition (33) should equal to zero, and (34) should be negative. However the solutions of the distributed algorithm produce error larger than those of the convex program, and the largest error is measured 2.19e-05. Moreover we observe that the errors produced by the solutions of the convex program are bounded well by the smallest precision bound (i.e., 4.572e-06), and all the KKT conditions given by (34) are satisfied with the solutions of the distributed algorithm as well as those of the convex program. Since the price dynamics in the distributed algorithm is asymptotic stable, it is natural to result in gaps and errors larger than those yielded by the convex program.

8. CONCLUDING REMARK

We investigate the market equilibrium in terms of Fisher model in multi-channel sharing CRN. PUs and SUs act as suppliers and purchasers, respectively: PUs offer their sub-channels to SUs with bounding the total amount of interference invoked from SUs’ transmissions, and SUs purchase the offered subchannels observing their budget constraints and the interference bounds given by the PUs. The utility functions of SUs and PUs are given as the achievable rates and the net profits, respectively.

The market equilibrium not only optimizes all traders’ (PUs and SUs) utilities but also achieves the market clearance. We show that the equilibrium is yielded by solving the optimization problem called Eisenberg-Gale convex program, and the equilibrium price is given by the Lagrangian dual variables of the convex program.

We also develop a distributed algorithm with which the traders can reach the equilibrium without any central authority. However, it is impossible to yield the exact equilibrium price since the system of the linear equations - that are composed of the KKT conditions of the convex program - are normally inconsistent, and the convergence behavior of the distributed algorithm is asymptotic. For these reasons, we provide the system of the linear equations with a certain precision bound that makes the system consistent, and give a termination threshold to the distributed algorithm. Moreover, we apply the adaptive step size to the price dynamics in the distributed algorithm for achieving higher convergence speed.

By the numerical experiments, we show that the solu-
tions achieved by the distributed algorithm are quite close to those of the convex program, and the errors in the KKT conditions are reasonably small. Besides, we show that the schemes with the adaptive step size are beneficial to achieve faster convergence.

9. REFERENCES


